3.5 Free (super) bosons

Let $\mathfrak{h}$ f.d superspace with non-degenerate supersym bilinear form: $(\cdot|\cdot)$

viewing $\mathfrak{h}$ as a commutative Liesuperalgebra

consider affinization

$$\hat{\mathfrak{h}} := \mathbb{C}[t,t^{-1}] \otimes_{\mathbb{C}} \mathfrak{h} + \mathbb{C}K$$

$$[a_m, b_n] = m(a|b)\,\delta_{m+n}\,K,$$

$$[K, \hat{\mathfrak{h}}] = 0 \qquad (a_n := t^n \otimes a)$$

The currents

$$a(z) = \sum_{n \in \mathbb{Z}} a_n z^{-n-1}, \quad a \in \mathfrak{h}$$

seen are mutually local (from Kostya's talk)

$$a(z)\, b(w) \sim \frac{(a|b)K}{(z-w)^2}$$

This is called Weyl affinization or just affinization to distinguish from next section Clifford affinization or super affinization Cl_A.

Def

$\mathfrak{g}$ Lie superalgebra
of formal distributions i.e spanned by mutually local $a^\alpha(z)$
a rep of $\mathfrak{g}$
is a field rep if
each $a^\alpha(z)$ is a field
for each $v \in V$ $a^\alpha_{(n)} v = 0$ $n >> 0$

Field rep of $\hat{h}$ on V,
then get set of mutually local fields with the previously written OPE which we call free (super) bosons
(super if $h_{\bar{1}} \neq 0$)
(sometimes this part is called symplectic fermion ?)

These satisfy conditions for Wick's theorem.

Choose basis $\{a^i\}$ $\{b^i\}$ of $\mathfrak{h}$

s.t. $(b^i|a^j) = \delta_{ij}$ consistent with $\mathbb{Z}_2$-grading

such bases are called dual.

Then for any $h \in \mathfrak{h}$

$$h = \sum_i (b^i|h) a^i$$

$$= \sum_i (a^i|h) b^i$$

Let

$$S(z) = \frac{1}{2} \sum_i :a^i(z) b^i(z):$$

Wick's theorem

$$S(z)\,\overline{a}(w) \sim \frac{1}{2}\sum_i \frac{(b^i|a)}{(z-w)^2}\,a^i(z)\,K$$

$$+\ \frac{1}{2}\sum_i (-1)^{p(b^i)}\,p(a)\,\frac{(a^i|a)}{(\cancel{z}-w)}\,b^i(z)\,\cancel{K}$$

using the basis expansion above

$$S(z)\,a(w) \sim \frac{a(z)}{(z-w)^2}\,K \sim \left(\frac{a(w)}{(z-w)^2} + \frac{\partial a(w)}{z-w}\right)K$$

Taylors formula

Let $K = k I_V$ affine central charge $k \neq 0$

$L(z) = \frac{1}{k} S(z)$

then

$L(z) a(w) \sim \frac{a(z)}{(z-w)^2} \sim \frac{a(w)}{(z-w)^2} + \frac{\partial a(w)}{z-w}$

$L(z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2}$

get

$[L_m, a_n] = -n a_{m+n}$

noting $L_0 = \frac{1}{2k} \sum_i a_0^i b_0^i + H$

$H = \frac{1}{2k} \sum_i \sum_{n>0} (a_{-n}^i b_n^i + (-1)^{p(a^i)} b_{-n}^i a_n^i)$

& a_0 lies in center of $\hat{h}$

$\Rightarrow [H, a_n] = -n a_n.$

i.e $[H, -]$ is a Hamiltonian

and all fields $a(z)$ have conformal weight 1.

in particular

$[L_{-1}, a(z)] = \partial a(z)$

$[L_0, a(z)] = (z\partial + 1) a(z)$

Since $L(z)$ is also local field by theorem from end of Kostya's talk we have that $L(z)$ is a Virasoro field, i.e L_n's satisfy Virasoro relations.

Next:

To compute central charge, have to look at $s=2$ terms of $L(z)L(w)$ in Wicks formula which is $\frac{1}{2}\frac{\operatorname{sdim}\mathfrak{h}}{(z-w)^4}$

$$\boxed{\text{central charge of } L(z) = \operatorname{sdim}\mathfrak{h}}$$

$\partial a(z)$ has conformal weight 2

$L^b(z) = L(z) + \partial b(z)$, $b \in \mathfrak{h}_{\bar{0}}$

local fields of conformal weight 2.

Then

$$L^b(z)a(w) \sim \frac{a(w)}{(z-w)^2} + \frac{\partial a(w)}{(z-w)} - \frac{2(a|b)k}{(z-w)^3}$$

$\Rightarrow$

$$[L^b_m, a_n] = -n a_{m+n} - (a|b)k(m^2+m)\,\delta_{m,-n}$$

$$[L^b_{-1}, a_n] = -n a_{n-1} \Rightarrow [L^b_{-1}, a(z)] = \partial a(z)$$

$\Rightarrow L^b(z)$ is Virasoro field.

by same calculation

central charge of $L^b(z)$ is $\underbrace{\dim \mathfrak{h}_{\bar{0}} - \dim \mathfrak{h}_{\bar{1}}}_{\operatorname{sdim}\mathfrak{h}} - 12(b|b)k$

rep theory of $\hat{\mathfrak{h}}$ $\underbrace{\hat{\mathfrak{h}}^< \qquad \hat{\mathfrak{h}}' \qquad \hat{\mathfrak{h}}^>}$

$$\hat{\mathfrak{h}} = t^0 \otimes \mathfrak{h} \oplus \Big(\bigoplus_{n<0} t^n \otimes \mathfrak{h} \Big) \oplus \mathbb{C}K \oplus \Big(\bigoplus_{n>0} t^n \otimes \mathfrak{h} \Big)$$

lie alg

study reps of $\hat{\mathfrak{h}}'$

Lemma

If v is a singular vector of field rep of $\hat{\mathfrak{h}}$ (i.e. $\hat{\mathfrak{h}}^> v = 0$)

then $Hv = 0$

Proof

def of H

$\boxtimes$

Let $\hat{\mathfrak{h}}^+ = \hat{\mathfrak{h}}^> + \mathbb{C}K$

Given $k \in \mathbb{C}$ denote π^k 1 dim rep of $\hat{\mathfrak{h}}^+$.

$$\pi^k(\hat{\mathfrak{h}}^>) = 0, \quad \pi^k(K) = k$$

Def

Verma module $\tilde{V}^k := \tilde{V}(\pi^k)$

$\tilde{V}^k = S(\hat{\mathfrak{h}}^<)$ symmetric superalgebra over $\hat{\mathfrak{h}}^<$

$K = kI$, $t^m \otimes a$ by multiplication if $m<0$

$$t^m \otimes a (t^{-n} \otimes b) = k m \delta_{m,n} (a|b) \quad n>0$$

if $m>0$

Thm

$\tilde{V}^k$ irreducible iff $k \neq 0$

and $\tilde{V}^0$ has unique maximal submodule

J^0 s.t $\tilde{V}^0/J^0$ is trivial 1-dim

Proof

If $k \neq 0$ then have H

and it is diagonilizable on $\tilde{V}^k$

with non negative eigenvalues,

Zero eigenvalues are multiples of 1

Hence by previous lemma $\tilde{V}^k$ irreducible

If $k \neq 0$

If $k = 0$ obvious.

$\boxtimes$

Example

$B := \tilde{V}^1$ is called oscillator rep

characterized by cyclic vector

$|0\rangle = 1 \in B$ i.e. $U(\hat{h}')|0\rangle = B$

s.t. $a_n|0\rangle = 0 \quad \forall n > 0$

Example

Oscillator algebra is special case of

$h = h_{\mathbb{C}} = \mathbb{C}$

$(a|b) = ab \quad a_n = 1_n$.

in this case $\tilde{V}^k$ can be identified

with $\mathbb{C}[x_1, x_2, \ldots]$

s.t

$$\alpha_m = \frac{\partial}{\partial x_m}, \quad \alpha_{-m} = kmx_m$$

$K = k \quad m > 0$

Free (super) fermion

Def

Let A f.d superspace with non degen anti-supersymmetric bilinear form $(-|-)$

Clifford affinization

$$C_A = \mathbb{C}[t, t^{-1}] \otimes_{\mathbb{C}} A + \mathbb{C}K$$

with $(m, n \in \frac{1}{2} + \mathbb{Z}, \varphi, \psi \in A$

$$[\varphi_m, \psi_n] = (\varphi|\psi)\delta_{m,-n}K, \quad [C_A, K] = 0$$

$$\varphi_n = t^{m-\frac{1}{2}} \otimes \varphi$$

field rep of C_A: V with $K = kI_V$ is called free (super) fermion if $A_{\bar{0}} \neq 0$

Like before these satisfy Wick's theorem.

$\{\varphi^i\}$ $\{\psi^i\}$ dual basis of A

$$L(z) = \frac{1}{2k}\sum_i :\partial\varphi^i(z)\psi^i(z): = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}$$

conf dim 2

$$L(z)\varphi(w) \overset{\text{Wick's}}{\sim} \frac{1}{2}\left(\frac{\varphi(z)}{(z-w)^2} + \frac{\partial\varphi(z)}{z-w}\right)$$

Same as before get $L(z)$ is Virasoro field

L_0 Hamiltonian

and central charge of $L(z) = -\frac{1}{2}\dim A$

We also get triangular decomp

$$C_A = C_A^{<} + \underbrace{\mathbb{C}K + C_A^{>}}_{C_A^{+}}$$

$$\tilde{V}^k := \tilde{V}(\pi^k)$$

where π^k is 1 dim K as k

$C_A^{>}$ as 0.

i.e. $S(C_A^{<})$

Theorem

$\tilde{V}^k$ is irreducible iff $k \neq 0$

Example

$F := \tilde{V}^1$ is called spin representation

same as before e, cyclic vector $|0\rangle$

s.t $\varphi_n|0\rangle = 0 \quad n>0$

Briefly Bosonization

Suppose A is direct sum of isotropic (form vanish) subspaces $A^+ \oplus A^-$

Let $k=1$ bases $\{\varphi^{\pm i}\}$ of $A^{\pm}$

s.t $(\varphi^{+i}|\varphi^{-j}) = \delta_{ij}$

$$\alpha(z) = \sum_i :\varphi^{+i}(z)\varphi^{-i}(z):$$

conf weight 1.

Using Wick's + Taylor's

$$\alpha(z)\alpha(w) \sim \frac{-\operatorname{sdim} A^+}{(z-w)^2}$$

i.e $\alpha(z)$ is free boson with affine central charge $-\operatorname{sdim} A^+$

can construct

$$L^\lambda(z) = (1-\lambda)L^+(z) + \lambda L^-(z), \quad \lambda \in \mathbb{C}$$

$$L^{\pm}(z) = \sum_i :\partial\varphi^{\pm i}(z)\varphi^{\mp i}(z)$$

and can show $L^\lambda(z)$ is Virasoro field for each λ

with central charge

$$c_\lambda = (12\lambda^2 - 12\lambda + 2)\operatorname{sdim} A^+$$